Stochastic and Nonlinear Dynamic Systems, AM40SD - Coursework 2 (2012-13)

(**All questions carry equal marks**)

Total: 100 marks

Due - 03.01.2013 at 12 PM; to be submitted to MB133

1. The Boussinesq approximated Navier-Stokes' flows in an infinite channel of width $2d$, heated from below, are given by:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0, \\ \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} &= -(1/\rho_0)\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{u} + g\alpha T \hat{\boldsymbol{\imath}}, \\ \frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} T &= \kappa \boldsymbol{\nabla}^2 T. \end{aligned}$$

   (a) Derive the non-dimensional form of the Navier-Stokes' equations given above using the thickness layer d, the time scale d^2/κ and the temperature scale $\Delta T/R$ respectively (R is a constant).

   [10 marks]

   (b) Define the control parameter upon which the stability calculations can be based, and arrive at a form of the unique (non-dimensional) parameter emerging from the non-dimensionalisation that defines the properties of the fluid flow. Provide an alternative physical interpretation for that parameter.

   [7 marks]

   (c) Derive the basic flow and temperature distributions.

   [8 marks]

   [Q1: 25 marks]

2. One way to identify the accurate transition to turbulence is via the sequential bifurcation of the basic flow (given by, for example, the calculations of (1c) above) as dictated by the non-linear terms of the Navier-Stokes' equations, displayed in the previous question. The introduction of the following two operators, $\boldsymbol{\delta}$, $\boldsymbol{\epsilon}$, makes the study of the sequential transition of the basic flow amenable to simulations:

$$\boldsymbol{\delta}\phi = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \hat{\boldsymbol{k}}\phi),$$
$$\boldsymbol{\epsilon}\phi = \boldsymbol{\nabla} \times \hat{\boldsymbol{k}}\phi,$$

for the velocity field $\boldsymbol{u} = \boldsymbol{\delta}\phi + \boldsymbol{\epsilon}\psi$.

(a) Find the explicit form of the operators $\boldsymbol{\delta}, \boldsymbol{\epsilon}$.

[10 marks]

(b) Show that for this representation of $\boldsymbol{u}$ the equation of incompressibility, given in (1) by:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,$$

plays no further role in the calculations for establishing the bifurcation sequence.[1]

[5 marks]

(c) Introduce perturbations and derive the linearised Navier-Stokes equations for the problem given in question (1) above.

[10 marks]

[Q2: 25 marks]

3. Expand the non-linear terms, $\boldsymbol{\epsilon}\cdot((\tilde{u}\cdot\nabla)(\tilde{u}))$ and get as far as you can.

Answer this question using the following steps below. Progress as far as you can. Marks will be rewarded for:

(a) providing the definitions of $\delta_1, \delta_2, \delta_3$ and $\epsilon_1, \epsilon_2, \epsilon_3$. **[3 marks]**.

(b) characterising the vectors, $\boldsymbol{\epsilon}$ and $\boldsymbol{\delta}$ using the fact that, $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3)$ and $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_3)$. **[1 mark]**

(c) calculating the coordinates of the vector, $\tilde{u} = \boldsymbol{\delta}\phi + \boldsymbol{\epsilon}\psi = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$. **[2 marks]**

(d) calculating the operator, $(\tilde{u}\cdot\nabla)$ where $\nabla = (\partial_1, \partial_2, \partial_3)$ **[1 mark]**

(e) remembering that, $(\tilde{u}\cdot\nabla)$ is a scalar operator which is going to be applied onto the vector, $\tilde{u}$. Evaluate $(\tilde{u}\cdot\nabla)(\tilde{u})$. **[5 marks]**.

(f) Also using the definition of $\boldsymbol{\epsilon}$ and $(\tilde{u}\cdot\nabla)(\tilde{u})$, calculate

$$\boldsymbol{\epsilon}\cdot((\tilde{u}\cdot\nabla)(\tilde{u}))$$

In order to do this, please ensure,

- that you use simple notations, for example, $\partial_1\partial_3\phi$ (this indicates a differentiation of ϕ with respect to x and z).
- that you maintain the correct order of the derivatives. $\partial_1\psi\partial_3\phi$ defines a first ordered derivative of ψ with respect to x that is multiplied with another first ordered derivative of ϕ with respect to z.

[15 marks]

[Q3: 25 marks]

[1] This is a considerable simplification for the relevant simulations.

4. (a) Two drunkards stagger to their right or to their left in equal step sizes while moving on a one-dimensional path. If 'p' defines the probability of a step towards the right for each of these walkers and if they start their journey exactly at the same time from the same point, what is the probability that they will meet each other again after N steps? Assume that both drunkards take their steps simultaneously.

[10 marks]

(b) A one dimensional random walker starts from the origin and takes steps either to the right or to the left with equal probability. What is the probability that after taking (a total of) N steps the walker returns back to the origin? You need to separately discuss the two situations: a) N = even, b) N = odd.

[7 marks]

(c) A (one dimensional) random walker has taken n_r steps to the right and n_l steps to the left, all with equal step sizes of unity (in either direction). Starting with the expression $W(n_r, n_l, m)$ for the probability of finding the random walker at the position m on the path after (s)he has taken total of N steps, calculate the form of $W(n_r, n_l, m)$ for $p << 1$, $n_r << N$ and N finite, where p = probability of a rightward step. Explain your result physically.

[8 marks]

[Hint: Both (correct) analytical and numerical solutions will carry the same mark.]

[Q4: 25 marks]

[TOTAL: 100 marks]