Stochastic and Nonlinear Dynamic Systems,

Global Bifurcation Analysis and Transition to Turbulence in Shear Incompressible Flow.

Notes - Part A

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1 Vertical Channel Internally Heated

1.1 Linear Analysis

We consider a homogeneously heated fluid between two infinite insulating boundaries and where the mass flux of the fluid flow accross any lateral cross section section of the channel is assumed constant. The(viscous) fluid is assumed incompressible. In this note we investigate the background that is required in order to attempt to obtain non-linear solutions for a vertical channel, homogeneously heated with pr = 7. Previous results at the linear stability level, provided by Gershuni and Zhukovistksi [**?**], were reproduced in the preparation of these notes, acting as a benchmark for the results presented herein. This work also extends the work by Generalis and Nagata [**?**, **?**], where in [**?**] pr = 0 and a pressure gradient was present in order to analyse the effect of purely hydrodynamic transition to instability. In [**?**] temperature was included and the channel was inclined at varying angles, but the mass flux condiion was not taken into account. In this respect the basic velocity profile in [**?**] was analogous to a Pouseille flow profile with no inflection points within the channel. In this section we extend the work by Gershuni [**?**] by including a basic flow with dual inflection points as [**?**], but with a finite Prandtl number for a more realistic model, in order to analyse the effects of both hydrodynamic and thermal mechanisms on the stability of the system, where we maintain a constant flux condition, at the fully non-linear level. This will allow us to study the sequential transition to turbulence as we vary certain control parameters.

1.2 Problem Modeling

We begin by outlining the coordinate system used and the orientation axes in relation to the physical problem, see Fig.1, with the origin of the position vector **r** fixed at the mid channel of the fluid layer with x, y, z as cartesian co-ordinates in the streamwise, spanwise and normal directions (with unit vectors $\hat{\mathbf{i}}$,$\hat{\mathbf{j}}$,$\hat{\mathbf{k}}$, respectively).

We employ the Boussinesq approximation and also include $\nabla\Pi$ (the pressure gradient) [1-3] in order to obtain the following Navier-Stokes equations for the velocity vector **u** and the temperature variation T:

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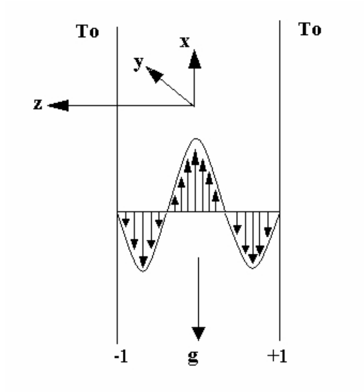

Figure 1: *Model Geometry.*

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla \Pi + g \gamma T \hat{\mathbf{i}} + \nu \nabla^2 \mathbf{u} \tag{1}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T + q \tag{2}$$

$$\nabla \cdot \mathbf{u} = 0 \quad , \tag{3}$$

$$\int_{-1}^{1} \mathbf{u} dz = 0 \tag{4}$$

$$\mathbf{u} = 0 \ , \ T = 0 \quad at \quad z = \pm 1 \tag{5}$$

Eqns. (1) and (2) are derived in Appendix 1. An outline of the use of the Boussinesq approximation and proof of the incompressibility condition (3) is dealt with at a later stage. The boundary conditions are reflected by eq.(5), where for convenience we have set $T_0 = 0$ for the fixed temperature of the boundaries, and we emply the no-slip conditions at the boundaries for the velocity field. These boundaries are readily modified for diverse models, making computations particularly easy to implement. We use the narrow gap approximation which enables us to employ the use of cartesian geometry and proceed to non-dimensionalize our system of equations. The derivation of the basic flow and temerature profiles using the stated boundary conditions will be provided later on.

The non-dimensionalisation of both the momentum and the temperature equations must be done so that the number of variables are reduced to a few manageable parameters. This is also needed for us to ignore scalars in the model which makes it more applicable.

For the non-dimensional description of the problem, we use the following parameters:

- d for length,
- $\frac{d^2}{\nu}$ for time,
- $\frac{\nu}{d}$ for velocity field($\mathbf{u}$),
- $\frac{1}{d}$ for ∇,
- $\frac{1}{d^2}$ for ∇^2,

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- $\frac{qd^2}{2\kappa Gr}$ for temperature,
- $\frac{g\alpha q d^5}{2\nu^2\kappa}$ for Gr, and
- $-\frac{d^3\nabla\pi}{2\nu^2\rho}$ for R.

The Grashof number gives the strength of the internal heat source. The Reynolds number $R = U_{max}d/\nu = -d^3\nabla\Pi/2\nu^2\rho$, measures the strength of the applied pressure gradient in the streamwise direction(U_{max} is the maximum laminar velocity), we adopt $R = 0$ in the current study.

After non-dimensionalisation we obtain the following Navier-Stokes equations for the velocity vector $\mathbf{u}$ and the temperature variation θ from the environment

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = 2R + T\hat{\mathbf{i}} + \nabla^2\mathbf{u} \quad (6)$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = -\frac{1}{Pr}(\nabla^2 T + 2Gr) \quad (7)$$

$$\nabla\cdot\mathbf{u} = 0 \quad (8)$$

Constant flux is maintained by applying eq.(4) which is dealt with in Appendix E as we need to find the integral of an infinite iterative equation in order to apply the mass flux condition in our computer model., We linearize the N.S. and apply the boundary conditions as shown in Appendix C to give the basic flow and temperature:

$$U_o(z)\hat{\mathbf{i}} = \frac{Gr}{60}(5z^4 - 6z^2 + 1) \quad (9)$$

$$T_o(z) = Gr(1 - z^2). \quad (10)$$

The Navier Stokes equations were also derived by Gershuni and Zhukhovitskii, who subsequently investigated the linear stability of the basic flow

$$\mathbf{u}_0(z) = \frac{Gr}{60}(5z^4 - 6z^2 + 1) \quad (11)$$

This basic steady state is derived by assuming that the total mass flux vanishes across any lateral cross- section of the channel. In comparison and for this work, we have made the assumption that the remote ends of the channel are open (as in Gershuni's work) and therefore our calculations assume the presence of a constant vertical pressure gradient. In the next section we need to derive the perturbation equations for linear stability analysis employing the Helmholz Decomposition of our solenoidal velocity field.

1.3 Perturbation Equations

We now separate the velocity deviations $\hat{\mathbf{u}}$ from the basic flow $\mathbf{u}_0(z)\hat{i}$ and the temperature deviations θ from the basic temperature $T_0(z)$ into average parts (over the x and y coordinates) $\breve{U}(z,t) \equiv \bar{\hat{\mathbf{u}}}$ and $\breve{T}(z,t) \equiv \bar{\theta}$ and a fluctuating part $\breve{\mathbf{u}}$, $\breve{\theta}$ respectively:

$$\hat{\mathbf{u}} = \breve{U}(z,t)\hat{\mathbf{i}} + \breve{\mathbf{u}} \quad (12)$$

$$\hat{\theta} = \breve{T}(z,t) + \breve{\theta}. \quad (13)$$

Where the average, indicated by the overbar, is obtained by applying $((\alpha\beta/4\pi^2)\int_0^{2\pi/\alpha}\int_0^{2\pi/\beta} dxdy)$. Further, we separate the soleniodal field $\breve{\mathbf{u}}$ into the poloidal and toroidal parts ϕ, ψ, by applying the operators $\delta_i = (\nabla\times(\nabla\times\mathbf{k}\cdot))_i$ and $\varepsilon_i = (\nabla\times(\mathbf{k}\cdot))_i$. Employing δ_i and ε_i

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assures that the incompressibility condition is satisfied automatically, thus playing no further part in the calculations. Our perturbation equations become

$$\frac{\partial}{\partial t}\nabla^2\Delta_2\phi - \nabla^4\Delta_2\phi + \hat{U}\partial_x\nabla^2\Delta_2\phi - (\partial_z^2\hat{U})(\partial_x\Delta_2\phi)$$
$$-\partial x\partial z\theta = \delta\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) \quad (14)$$
$$\frac{\partial}{\partial_t}\Delta_2\psi - \partial_y\theta - \nabla^2\Delta_2\psi - (\partial_z\hat{U})(\partial_y\Delta_2\phi) + \hat{U}\partial_x\Delta_2\psi$$
$$= \varepsilon\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) \quad (15)$$

while we can rewrite the temperature equation in the form

$$\frac{\partial}{\partial t}\theta = -2Gr(\mathbf{r}\cdot\mathbf{k})\Delta_2\phi - \hat{U}\partial_x\theta + \Delta_2\phi\partial_z\check{T} + \frac{1}{Pr}\nabla^2\theta - (\delta\phi + \varepsilon\psi)\cdot\nabla\theta, \quad (16)$$

where we have dropped the ˇ from the temperature fluctuations and $\Delta_2 \equiv \partial_x^2 + \partial_y^2$ is the planform Laplacian. $\delta\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$ and $\varepsilon\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$ are the nonlinear parts of the Navier-Stokes Equations and are not decomposed until section 5 where we begin the non-linear analysis. The mean flow and mean temperature, $\check{U}(z,t)$ and $\check{T}(z,t)$, satisfy

$$\partial_z^2\check{U} + \check{T} + \partial_z\overline{\Delta_2\phi(\partial_x\partial_z\phi + \partial_y\psi)} = \partial_t\check{U}, \quad (17)$$
$$\partial_z^2\check{T} + Pr\partial_z\overline{(\Delta_2\phi)\theta} = Pr\partial_t\check{T}. \quad (18)$$

Eqns.(14-18) are subject to the homogeneous boundary conditions

$$\phi = \partial\phi/\partial z = \check{U} = \psi = \check{T} = \theta = 0 \qquad at\ z = \pm 1. \quad (19)$$

2 Appendix 1 - The Navier-Stokes Equations

Momentum Equation

Assume the fluid between the plates is in motion and the velocity field, **u** (x,y,z,t) is in space. The rate of change of **u** at some point (x,y,z) is $\frac{\partial \mathbf{u}}{\partial t}$. As we are following the fluid between the plates, we therefore use the Material Derivative, $\frac{D\mathbf{U}}{Dt}$, where

$$\frac{D\mathbf{U}}{Dt} = \frac{d}{dt}\mathbf{u}[x(t),y(t),z(t),t], \quad (20)$$

and the points x(t), y(t) and z(t) change with time as velocity changes with time. Writing the rates of changes as $\mathbf{u} = \frac{\partial x}{\partial t}$, $\mathbf{v} = \frac{\partial y}{\partial t}$, $\mathbf{w} = \frac{\partial z}{\partial t}$, we can express the material derivative as

$$\frac{D\mathbf{u}}{Dt} = \Delta\mathbf{u}(x,t) + \Delta\mathbf{u}(y,t) + \Delta\mathbf{u}(z,t) + \Delta\mathbf{u}(t), \quad (21)$$

where Δ (delta) represents the changes in each variable.

Applying the chain rule:

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial\mathbf{u}}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial\mathbf{u}}{\partial z}\cdot\frac{\partial z}{\partial t} + \frac{\partial\mathbf{u}}{\partial t} \quad (22)$$

gives

$$\frac{D\mathbf{u}}{Dt} = u\frac{\partial\mathbf{u}}{\partial x} + v\frac{\partial\mathbf{u}}{\partial y} + w\frac{\partial\mathbf{u}}{\partial z} + \frac{\partial\mathbf{u}}{\partial t}. \quad (23)$$

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Equation [23] can be written using the grad ∇ operator as

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \tag{24}$$

We now need to define the forces acting on the fluid.

There is always a resultant force acting on every volume of the fluid.The force comprises of acceleration due to gravity and pressure due to interaction with surroundings, acting on a fluid mass, as shown in figure 1.1.

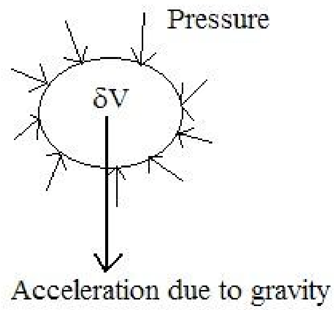

Figure 2: *Forces acting on a volume of fluid*

So, the resultant force which is simply the force per unit volume, is expressed as (-∇p + ρg), where p = pressure, ρ = density of the fluid, g = acceleration due to gravity. From Newton's second law **F** = ma, we can write $(-\nabla p + \rho g)\delta V = \rho \delta V \cdot \frac{D\mathbf{u}}{Dt}$ because ρδ V (density x volume) is the mass of the fluid and the acceleration of a fluid particle is described as the material derivative.

As δV is not equal to 0, we can divide by δV, giving

$$-\nabla p + \rho g = \rho \frac{D\mathbf{u}}{Dt} \tag{25}$$

Now dividing by ρ, we can express the material derivative as

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} + g. \tag{26}$$

Substituting equation [26] into equation [24], we obtain

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + g. \tag{27}$$

As we are using a viscous fluid, there is another force called the viscous force acting on the fluid. This viscous force is usually described in terms of the shearing (τ) and the normal or strain (σ) stress on the fluid particles.

Stress and strain tensors can be derived as a direct result of the fluid particles moving against each other:

$$-\frac{\nabla p}{\rho} + g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \frac{\partial \mathbf{u}}{\partial t} + u\frac{\partial \mathbf{u}}{\partial x} + v\frac{\partial \mathbf{u}}{\partial y} + w\frac{\partial \mathbf{u}}{\partial z}, \tag{28}$$

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which implies that

$$-\frac{\nabla p}{\rho}+g_x+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}=\rho\frac{\partial\mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\nabla)\mathbf{u}. \quad (29)$$

For incompressible Newtonian fluids, the stresses are linearly related to the rates of deformation and can be expressed as:

for normal stress: $\sigma_{xx}=-p+2\nu\frac{\partial\mathbf{u}}{\partial x}$

for shearing stress: $\tau_{yx}=\nu(\frac{\partial\mathbf{u}}{\partial y}+\frac{\partial v}{\partial x})$,

$\tau_{zx}=\nu(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z})$,

where ν is the kinematic viscosity.

Applying the stresses to the equation of motion [29], gives

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{\nabla p}{\rho}+g+\nu\nabla^2\mathbf{u}. \quad (30)$$

Now, we need to apply the Boussinesq Approximation as the fluid has its own heat source (like a thermos flask or nuclear fuel rod). Chandrasekhar [**?**] gives a clear overview of the application of this approximation. The Boussinesq approximation states that as the fluid moves from cold region to hot region, the density (ρ) remains constant except for the effect of gravity acting on the density. If we write $(T-T_0)$ as T, in the equation of states ($\rho=\rho_0[1-\alpha(T-T_0)]$), we get

$$\rho=\rho_0-\rho_0\alpha T, \quad (31)$$

which is same as

$$g(\rho_0-\rho)=-g\alpha T, \quad (32)$$

where α is the coefficient of volume expansion. For liquids, α is in the range $10^{-}3$ to $10^{-}4$.
As the fluid is moving upwards against gravity and in the $\hat{\mathbf{i}}$ direction, we therefore write -gα $T\hat{\mathbf{i}}$.

Hence, equation [30] becomes

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}-g\alpha T\hat{\mathbf{i}}=-\frac{\nabla p}{\rho}+\nu\nabla^2\mathbf{u}. \quad (33)$$

By rearranging the terms, we get

$$\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\frac{\nabla p}{\rho}+g\alpha T\hat{\mathbf{i}}+\nu\nabla^2\mathbf{u}. \quad (34)$$

where $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is the nonlinear advection term, $-\frac{\nabla p}{\rho}$ is the pressure component, $g\alpha\,T\hat{\mathbf{i}}$ is the buoyancy term, $\nu\nabla^2\mathbf{u}$ is the viscous force;

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Energy Equation

Gauss's Theory is useful. Energy of the fluid is expressed in this case as the sum of the absolute thermodynamic internal energy per unit mass, e, and the kinetic energy per unit mass, $\frac{1}{2}v^2$, where v is the magnitude of the velocity vector.

The change in total energy per unit volume of the fluid in the control volume is $\frac{\partial}{\partial t}[\rho(e+\frac{1}{2}v^2)]$. If we replace $[\rho(e+\frac{1}{2}v^2)]$ by T, we can express the change in total energy in a simpler way as $\frac{\partial T}{\partial t}$.

We now apply the Fourier's law of heat conduction that relates to the heat flow in the z-direction, to the rate of change of temperature in the z-direction, which is expressed as

$$q_z = -\kappa_z A \frac{\partial T_f}{\partial z_k}, \tag{35}$$

where q_z is the heat flux, κ_z is heat conduction coefficient in the z-direction, A represents the surface area perpendicular to the z-direction,and T_f represents the temperature of the flow. If heat flow is considered positive when flowing from the control volume to the surroundings, this means that q_z in is negative. Therefore, for q_z to be positive, we need the minus sign. Thus, the total heat transferred into the control volume is

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z}. \tag{36}$$

Thus, by applying equation [36] to equation [35], the heat flow rate per unit volume is

$$\frac{\partial}{\partial x}(\kappa_x \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(\kappa_y \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(\kappa_z \frac{\partial T}{\partial z}), \tag{37}$$

which in the case of constant heat conduction coefficient $(\kappa_x = \kappa_y = \kappa_z)$ becomes

$$\kappa(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}) = \kappa \nabla^2 T. \tag{38}$$

There is also rate of work per unit volume being done by the surface forces found by multiplying the stress by the gravity force vector.

If we call these forces q, we get the temperature equation as

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T + q, \tag{39}$$

where q is the volume strength of the heat source that generates the basic flow.

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3 Appendix 2 - Derivation of the Perturbation Equations

3.1 Preliminary Notes on Tensor Algebra

Notes on Tensors

For more information consult Kendall's[?] and Aris's book[?].

1. Permutation Tensor (Anti-symmetric)

$\varepsilon_{ijk} = 1$ if ijk is cyclic 123, 312 etc. $\varepsilon_{ijk} = -1$ if ijk is anticyclic 321,132 etc. and $\varepsilon_{ijk} = 0$ if any two of i,j or k are equal.

The anti-symmetric rule states that : $\varepsilon_{ijk}\varepsilon_{jmn} = -\varepsilon_{jik}\varepsilon_{jmn}$.

2. The Kronecker Delta (Symmetric)

$\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Repeated indices are assumed summed over. Therefore $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$.

The Kronecker Delta and Permutation Tensor sare related by:

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} = \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{lm}) - \delta_{im}(\delta_{jl}\delta_{ln} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{lk}). \quad (40)$$

1. One Index contracted ($i = l$) and equation 40 becomes: $\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, hence we obtain a widely used identity;

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \quad (41)$$

2. Two indices contracted ($i = l, j = m$) and equation 41 becomes $2\delta_{kn}$:

$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn} \quad (42)$$

3. Three Indices contracted ($i = l, j = m, k = n$) and equation 42 becomes:

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6. \quad (43)$$

Some more examples follow (λ_i the coordinates of $\hat{k}$):

1. $\varepsilon_{jlm}\partial_j\lambda_l\partial_m\psi \;\; \varepsilon_{ipq}\lambda_i\lambda_p\partial_q\partial_t^2\psi = 0$.

2. $\lambda_i\lambda_i = \lambda_1\lambda_1 + \lambda_2\lambda_2 + \lambda_3\lambda_3 = 0 + 0 + 1 = 1$, where $\{\lambda_i\} = \hat{\mathbf{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

3. $(\lambda_i\lambda_j\partial_j\Delta\phi)\left(\lambda_i\partial_j\partial_t^2\Delta\phi\right) = (\Delta\phi)\left(\lambda_j\partial_j\partial_t^2\Delta\phi\right) = (\Delta\phi)\left(\partial_z\partial_t^2\Delta\phi\right)$.

4. If $\partial_i\varepsilon_{ipq}\lambda_p\partial_q\partial_z\psi \cdot \partial_j\varepsilon_{jlm}\lambda_l\partial_m\psi$, then this expression should be recognized as $(\nabla\cdot\hat{k}\times\nabla\partial_z\psi) \quad (\nabla\cdot\hat{k}\times\nabla)\psi$.

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5. $\varepsilon_{i3q}\varepsilon_{j3m} = [\delta_{ij}\delta_{qm} - \delta_{im}\delta_{qj}]$

6. $\varepsilon_{ipq}\lambda_p\partial_q\partial_j\partial_z\partial_i\psi = 0$

7. $\varepsilon_{jlm}\cdot\varepsilon_{ipq}\lambda_l\lambda_p\partial_m\partial_q\partial_j\partial_i\partial_z\psi = 0.$

3.2 Derivation of explicit expressions for the vector operators ε_i and δ_i

For the solenoidal fluid velocity field u we write:

$$u = \delta\phi + \varepsilon\psi = \nabla\times\nabla\times\phi\hat{k} + \nabla\times\psi\hat{k}\ ,$$

where ϕ, ψ are the poloidal and toroidal components of the velocity field respectively. We shall derive the curl curl operator expression firstly, as it involves the poloidal component of u. In the following we derive expressions in the conventional way and show the much more elegant expressions involving δ and ε. Sequentially we have:

$$\nabla\times\hat{k}\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \phi \end{vmatrix} = \hat{i}(\partial_y\phi) - \hat{j}(\partial_x\phi) + \hat{k}(0). \tag{44}$$

Taking the curl again,

$$\nabla\times\nabla\times\hat{k}\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_y\phi & -\partial_x\phi & 0 \end{vmatrix} = \hat{i}(\partial_x\partial_z\phi) + \hat{j}(\partial_y\partial_z\phi) + \hat{k}(-\partial_x^2 - \partial_y^2)\phi. \tag{45}$$

We can therefore make the association:

$$u_i = (\partial_i\partial_z - \lambda_i\Delta)\phi, \tag{46}$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ and $\lambda_i\cdot\Delta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\cdot(\partial_x\partial_y\partial_z)$, $\lambda_x = \lambda_y = 0$ with $\lambda_z = 1$.
We define $\delta_i = \partial_i\partial_z - \lambda_i\Delta$ as our curl curl operator, with components: $i = 1 \rightarrow \partial_x\partial_z - 0$, $i = 2 \rightarrow \partial_y\partial_z - 0$ and $i = 3 \rightarrow \partial_z^2 - (\partial_x^2 + \partial_y^2 + \partial_z^2) = \Delta_2$, the planform Laplacian.

Now let us derive our curl operator for the toroidal component of the velocity field:

$$\nabla\times\hat{k}\psi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi \end{vmatrix} = -\hat{i}(\partial_y\psi) + \hat{j}(\partial_x\psi) + \hat{k}(0). \tag{47}$$

Using the permutation tensor outlined in the previous section in this appendix we set:

$$\varepsilon_i = \varepsilon_{ijk}\lambda_j\partial_k\psi. \tag{48}$$

Working through using values of i = 1,2,3=x,y,z respectively:
$i = 1 \rightarrow \varepsilon_{123}\lambda_2\partial_z + \varepsilon_{132}\lambda_3\partial_y = -\partial_y$,

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$i = 2 \rightarrow -\varepsilon_{213}\lambda_1\partial_z + \varepsilon_{231}\lambda_2\partial_x = \partial_x,$
$i = 3 \rightarrow -\varepsilon_{321}\lambda_2\partial_x + \varepsilon_{312}\lambda_1\partial_y = 0.$

We then can define $\varepsilon_i = \varepsilon_{ijk}\lambda_j\partial_k$ as our curl operator.

3.3 Proof of Incompressibilty Condition

$$\begin{aligned} 0 &= \nabla \cdot u \\ &= \nabla \cdot (\delta\phi + \varepsilon\psi) \\ &= \partial_x(\delta\phi + \varepsilon\psi)_x + \partial_y(\delta\phi + \varepsilon\psi)_y + \partial_z(\delta\phi + \varepsilon\psi)_z \\ &= \partial_i[\partial_i\partial_z\phi - \lambda_i\Delta\phi + \varepsilon_{ijk}\lambda_j\partial_k\psi] \\ &= \partial_i\partial_i\partial_z\phi + \varepsilon_{ijk}\lambda_j\partial_i\partial_k\psi - \lambda_i\partial_i\Delta\phi \\ &= 0. \end{aligned} \tag{49}$$

3.4 Curl and curl curl of the motion equation

3.4.1 Curl of the motion equation

We take the non-dimensionalised motion equation and apply the ε operator on each part separately;

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = 2R + T\hat{\mathbf{i}} + \nabla^2\mathbf{u}. \tag{50}$$

For convenience I shall drop the use of the bold font.

1. $\varepsilon \cdot u$:

$$\varepsilon_i u_i = \varepsilon_i(\delta_i\phi + \varepsilon_i\psi) = \varepsilon_i\delta_i\phi + \varepsilon_i\varepsilon_i\psi \tag{51}$$

In this equation, $\varepsilon_i\delta_i\phi = 0$. This is because ε_i and δ_i are orthogonal to each other, and also

$$\begin{aligned} \delta_i\varepsilon_i &= (\partial_i\partial_z - \lambda_i\Delta)\varepsilon_{ijk}\lambda_j\partial_k \\ &= \partial_i\partial_z E_{ijk}\lambda_j\partial_k - \Delta\varepsilon_{ijk}\lambda_i\lambda_j\partial_k. \end{aligned} \tag{52}$$

However, λ_i, λ_j andλ_k are always $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Therefore, $i = j$ and therefore according to the property of permutation tensor $\varepsilon_{ijk}\lambda_i\lambda_j\partial_k$ will be 0 due to the fact that two indices are the same.

Also, if we fix $j = 3$ in $\partial_i\partial_z\varepsilon_{ijk}\lambda_j\partial_k$, we get $\partial_i\partial_z\varepsilon_{i3k}\lambda_j\partial_k$ which is

$$\begin{aligned} \varepsilon_{i3k}\partial_i\partial_z\partial_k &= \varepsilon_{132}\partial_1\partial_z\partial_2 + \varepsilon_{231}\partial_2\partial_z\partial_1 \\ &= -\partial_x\partial_z\partial_y + \partial_y\partial_z\partial_z \\ &= 0. \end{aligned} \tag{53}$$

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Therefore,

$$\begin{aligned}
\varepsilon_i u_i &= \varepsilon_i \varepsilon_i \psi \\
&= \varepsilon_{ijk}\lambda_j \partial_k \varepsilon_{ilm}\lambda_l \partial_m \psi \\
&= \varepsilon_{ijk}\varepsilon_{ilm}\lambda_j \lambda_l \partial_k \partial_m \psi \\
&= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\lambda_j \lambda_l \partial_k \partial_m \psi \\
&= \delta_{jl}\delta_{km}\lambda_j \lambda_l \partial_k \partial_m \psi - \delta_{jm}\delta_{kl}\lambda_j \lambda_l \partial_k \partial_m \psi
\end{aligned} \tag{54}$$

Now, for δ_{jl} to be 1, j must equal to l. Similarly, $k = m$ for $\delta_{km} = 1$, $j = m$ for $\delta_{jm} = 1$, and $k = l$ for $\delta_{kl} = 1$.
Therefore, $\varepsilon_i u_i = \lambda_j \lambda_j \partial_k \partial_k \psi - \lambda_j \partial_j \lambda_l \partial_l \psi$.
But, $\lambda_j \partial_j = \lambda_l \partial_l = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \partial_z$, and $\lambda_j \lambda_j = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1$.
Therefore,

$$\begin{aligned}
\varepsilon_i u_i &= \partial_k^2 \psi - \partial_z^2 \psi \\
&= (\partial_x^2 + \partial_y^2 + \partial_z^2)\psi - \partial_z^2 \psi \\
&= \partial_x^2 \psi + \partial_y^2 \psi + \partial_z^2 \psi - \partial_z^2 \psi \\
&= \partial_x^2 \psi + \partial_y^2 \psi \\
&= (\partial_x^2 + \partial_y^2)\psi
\end{aligned} \tag{55}$$

Here, we can apply the Planform Laplacian, Δ_2. Thus, $\varepsilon_i u_i = \Delta_2 \psi$.

2. The Reynolds number is a constant. Therefore, when applying the curl on R will give zero.So,

$$\begin{aligned}
\varepsilon \cdot 2R &= \varepsilon_i R_i \\
&= \varepsilon_{ijk}\lambda_j \partial_k R \\
&= 0
\end{aligned} \tag{56}$$

3. The curl on temperature will effectively give zero because temperature is a scalar.

$$\begin{aligned}
\varepsilon \cdot T\hat{i} &= \varepsilon_i T\hat{i} \\
&= \varepsilon_{ijk}\lambda_j \partial_k T\hat{i} \\
&= 0
\end{aligned} \tag{57}$$

4. The curl on $\varepsilon \cdot \nabla^2 u$ is simply the curl on u because the curl on ∇^2 is ∇^2 is itself. And, the curl on **u** was already found out in (1). So,

$$\begin{aligned}
\varepsilon \cdot \nabla^2 u &= \varepsilon \cdot \nabla^2 (\varepsilon \cdot u) \\
&= \nabla^2 \delta_2 \psi.
\end{aligned} \tag{58}$$

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5. The curl of the linear components of $u\cdot\nabla u$.

   The velocity field, u is comprised of 2 components:
   $u = u_0(z)\hat{i} + \breve{u}$,
   where

   - $u_0(z)$ is the basic flow, and
   - $\breve{u}$ is the deviation.

   Further, $\breve{u} = \hat{u} + \bar{u}$, where

   - $\hat{u}$ is the perturbation, and
   - $\bar{u}$ is the mean flow.

   So,

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$$u\cdot\nabla u = (u_0(z)\hat{i} + \hat{u} + (u_0(z)\hat{i} + \hat{u} + \bar{u}) \tag{59}$$

giving $u_0(z)\hat{i}\cdot\nabla u_0(z)\hat{i}+$ other terms which may be linear in perturbations.
i.e:

$$u\cdot\nabla u = \bar{u}\cdot\nabla\breve{u} + \breve{u}\cdot\nabla\bar{u} + \breve{u}\cdot\nabla\breve{u}, \tag{60}$$

where $\breve{u}\cdot\nabla\breve{u}$ is the non-linear term, which we ignore because my system is linear.

The curl of $(u\cdot\nabla)u$:

$$\varepsilon\cdot(u\cdot\nabla u) = \varepsilon\cdot(\bar{u}\cdot\nabla\breve{u}) + \varepsilon\cdot(\breve{u}\cdot\nabla\bar{u}) \tag{61}$$

$$\begin{aligned}
\varepsilon\cdot(\bar{u}\cdot\nabla\breve{u}) &= \varepsilon_i(\bar{u}\cdot\nabla u_i)\\
&= \varepsilon_{ijk}\lambda_k\partial_j[\nabla(\partial_i\partial_z - \lambda_i\Delta)\phi - \varepsilon_{ilm}\lambda_m\partial_l\psi]\bar{u}\\
&= \varepsilon_{ijk}\lambda_k\partial_j[\partial_i\nabla\partial_z\phi - \nabla\Delta\lambda_i\phi - \varepsilon_{ilm}\lambda_m\partial_l\nabla\psi]\bar{u}\\
&= [\varepsilon_{1j3}\partial_j\partial_i\nabla\partial_z\phi - \varepsilon_{3j3}\partial_j\nabla\Delta\lambda_i\lambda_k\phi - \varepsilon_{ljk}\varepsilon_{ilm}\lambda_k\lambda_m\partial_j\partial_l\nabla\psi]\bar{u}\\
&= 0 - 0(-\varepsilon_{ijk}\varepsilon_{ilm}\lambda_k\lambda_m\partial_j\partial_l\nabla\psi)\bar{u}\\
&= -(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\lambda_k\lambda_m\partial_j\partial_k\nabla\psi\bar{u}\\
&= (-\nabla^3\psi + \nabla\partial_z^2\psi)\bar{u}\\
&= \Delta_2\nabla\bar{u}\psi\\
&= \bar{u}\cdot\nabla\Delta_2\psi
\end{aligned} \tag{62}$$

$$\begin{aligned}
\varepsilon\cdot(\breve{u}\cdot\nabla\bar{u}) &= \varepsilon_i(u_j\cdot\nabla\bar{u})_i\\
&= \varepsilon_{ipq}\lambda_q\partial_p(\breve{u}_j\cdot\nabla_j u_0(z))\\
&= \varepsilon_{123}\partial_p(\breve{u}_j\cdot\nabla_j u_0(z))\\
&= \partial_y(\breve{u}_j\cdot\partial_j u_0(z))
\end{aligned} \tag{63}$$

Here, $\breve{u}_j\cdot\partial_j u_0(z) = (u_x\partial_x + u_y\partial_y + u_z\partial_z)u_0(z)$. But, we want only the z-component. Therefore,

$$\begin{aligned}
\varepsilon\cdot(\breve{u}\cdot\nabla\bar{u}) &= \partial_y(\partial_j\partial_z - \lambda_j\Delta)\phi u_0(z)\\
&= \partial_y(\partial_{z^2} - \Delta)\phi u_0(z)\\
&= \partial_y\Delta_2\phi u_0(z)
\end{aligned} \tag{64}$$

Therefore,

$$\varepsilon\cdot(u\cdot\nabla u) = \bar{u}\cdot\nabla\Delta_2\psi + \partial_y\Delta_2\phi u_0(z) \tag{65}$$

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3.4.2 The curl curl of the motion equation

Let us take the double curl of each component individually.

1.

$$\begin{aligned}\delta\cdot\mathbf{u} &= \delta_i u_i\\ &= \delta_i(\delta_i\phi+\varepsilon_i\psi)\\ &= \delta_i\delta_i\phi+\delta_i\varepsilon_i\psi\\ &= \delta_i\delta_i\phi \quad (\delta_i \text{ is orthogonal to } \varepsilon_i \text{ and thus, } \delta_i\varepsilon_i\psi \text{ goes to zero})\\ &= (\partial_i\partial_z-\lambda_i\Delta)(\partial_i\partial_z-\lambda_i\Delta)\phi\\ &= (\partial_i^2\partial_z^2-2\partial_i\partial_z\lambda_i\Delta+\lambda_i^2\Delta^2)\phi\end{aligned} \tag{66}$$

Now, $\partial_i^2=\partial_x^2+\partial_y^2+\partial_z^2=\Delta$, $\partial_i\lambda_i=\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}=\partial_z$, and $\lambda_i^2=\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}=1$. Therefore,

$$\begin{aligned}\delta\cdot\mathbf{u} &= (\Delta\partial_z^2-2\partial_z^2\Delta+\Delta^2)\phi\\ &= (\Delta^2-\partial_z^2\Delta)\phi\\ &= \Delta(\partial_x^2+\partial_y^2+\partial_z^2-\partial_z^2)\phi\\ &= \Delta(\partial_x^2+\partial_y^2)\phi\\ &= \Delta\Delta_2\phi\\ &= \nabla^2\Delta_2\phi\end{aligned} \tag{67}$$

2.

$$\begin{aligned}\delta\cdot 2R &= \delta_i R_i\\ &= (\partial_i\partial_z-\lambda_i\Delta)R_i\\ &= 0 \quad \text{This is zero because R is a constant.}\end{aligned} \tag{68}$$

3.

$$\begin{aligned}\delta\cdot T\hat{\mathbf{i}} &= \delta_i T\hat{\mathbf{i}}\\ &= (\partial_i\partial_z-\lambda_i\Delta)\hat{\mathbf{i}}T\\ &= (\partial_i\partial_z\hat{\mathbf{i}}-\lambda_i\Delta_i)T\end{aligned} \tag{69}$$

Here, $\partial_i\partial_z=\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\partial_x$ and $\lambda_i\Delta_i=1$. Therefore,

$$\delta\cdot T\hat{\mathbf{i}}=(\partial_x\partial_z-\Delta)T. \tag{70}$$

4.

$$\delta\cdot\nabla^2\mathbf{u}=(\delta\cdot\nabla^2)(\delta\cdot\mathbf{u}) \tag{71}$$

But, $(\delta\cdot\nabla^2)$ gives ∇^2 and $(\delta\cdot\mathbf{u})$. Therefore,

$$\begin{aligned}\delta\cdot\nabla^2\mathbf{u} &= \nabla^2\nabla^2\Delta_2\phi\\ &= \nabla^4\Delta_2\phi\end{aligned} \tag{72}$$

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