New Results for the Couette-Taylor System in the Small Gap Limit

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Taylor Couette flow

- Fluid flow between two concentric cylinders rotating with different velocities ($r_1 \Omega_1 \neq r_2 \Omega_2$)

- Instability based on the centrifugal force ($r_1 \Omega_1^2 > r_2 \Omega_2^2$)

- Taylor vortex flow (monotonic in azimuthal direction)
States

Twisted vortex flow

Wavy inflow boundary

Wavy outflow boundary

Wavelets

Figs 4, 8, 9, 10 of Andereck et.al: Phys. Fluids, 26, 1395–1401 (1983)
Small gap approximation
for nearly co-rotating cylinders

$$\left| \frac{\Omega_2 - \Omega_1}{\Omega_2 + \Omega_1} \right| \ll 1, \quad \frac{r_2 - r_1}{r_2 + r_1} \ll 1$$

$$(r_1 \Omega_1 > r_2 \Omega_2)$$
Governing equation

- Navier-Stokes equation (normalised):
  \[
  \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \right] \mathbf{u} + \mathbf{\Omega} \times \mathbf{u} = -\nabla \Pi + \nabla^2 \mathbf{u},
  \]
  \[\nabla \cdot \mathbf{u} = 0\]

- Boundary condition:
  \[\mathbf{u} = \mp R \hat{\mathbf{i}} \quad \text{at} \quad z = \pm 1\]

- Basic profile:
  \[\mathbf{u} = -R z \hat{\mathbf{i}}\]

- Time scale: \(d^2/\nu\)
- Length scale: \(d\)
- Velocity unit: \(\nu/d\)
Control parameters

- Reynolds number:
  \[ R \equiv (\Omega_1 - \Omega_2) (r_1 + r_2) d/4\nu \]

- Rotation number (twice the mean rotation rate normalised):
  \[ \Omega = (\Omega_1 + \Omega_2) d^2/\nu \]

- Time scale: \( d^2/\nu \)
- Length scale: \( d \)
- Velocity unit: \( \nu/d \)

- Kinematic viscosity: \( \nu \)
- Half gap width:
  \[ d = \frac{r_2 - r_1}{2} \]
**Decomposition of flow field** $\mathbf{u}$

Elimination the equation of continuity:

$$ u = (-R z + U(t, z)) \hat{i} + V(t, z) \hat{j} + \tilde{\mathbf{u}}, $$

$$ \tilde{\mathbf{u}} = \nabla \times (\nabla \times k \phi) + \nabla \times k \psi. $$

$-R z$: basic flow

$U$: mean flow part in azimuthal ($x$) direction

$V$: mean flow in span-wise ($y$) direction

$\phi$: poloidal potential

$\psi$: toroidal potential

$\hat{i}$: unit vector in ($x$) direction

$\hat{j}$: unit vector in ($y$) direction

$k$: unit vector normal to wall ($z$) direction
Poloidal-Toroidal decomposition

Operators $k \cdot \nabla \times (\nabla \times \ldots,)$ and $k \cdot \nabla \times$ on Navier-Stokes equation we obtain the following two equations for $\phi$ and $\psi$:

Poloidal($\phi$) equation:

$$\left( \nabla^2 - \frac{\partial}{\partial t} \right) \nabla^2 \Delta_2 \phi - \Omega \frac{\partial}{\partial y} \Delta_2 \psi$$

$$= (-Rz + U) \frac{\partial}{\partial x} \nabla^2 \Delta_2 \phi - \frac{\partial^2 U}{\partial z^2} \frac{\partial}{\partial x} \Delta_2 \phi$$

$$- \frac{\partial^2 V}{\partial z^2} \frac{\partial}{\partial y} \Delta_2 \phi + V \frac{\partial}{\partial y} \nabla^2 \Delta_2 \phi + k \cdot \nabla \times (\nabla \times (\tilde{u} \cdot \nabla \tilde{u}))$$

Toroidal($\psi$) equation:

$$\left( \nabla^2 - \frac{\partial}{\partial t} \right) \Delta_2 \psi + \Omega \frac{\partial}{\partial y} \Delta_2 \phi$$

$$= (-Rz + U) \frac{\partial}{\partial x} \Delta_2 \psi + \left( R - \frac{\partial U}{\partial z} \right) \frac{\partial}{\partial y} \Delta_2 \phi$$

$$+ V \frac{\partial}{\partial y} \Delta_2 \psi + \frac{\partial V}{\partial z} \frac{\partial}{\partial x} \Delta_2 \phi - k \cdot \nabla \times (\tilde{u} \cdot \nabla \tilde{u})$$
Mean flow equations

Two equations for the mean flows in the azimuthal $U$ and axial $V$ directions by taking average over $(x, y)$ plane:

\[
\left( \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t} \right) U = - \frac{\partial}{\partial z} \left\langle \Delta_2 \phi \left( \frac{\partial^2}{\partial x \partial z} \phi + \frac{\partial}{\partial y} \psi \right) \right\rangle
\]

\[
\left( \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t} \right) V = - \frac{\partial}{\partial z} \left\langle \Delta_2 \phi \left( \frac{\partial^2}{\partial y \partial z} \phi - \frac{\partial}{\partial x} \psi \right) \right\rangle
\]

Herein $\Delta_2 \equiv \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$, the average over $(x, y)$:

\[
\left\langle \cdot \right\rangle \equiv \frac{\alpha \beta}{4\pi^2} \int_0^{2\pi/\alpha} \int_0^{2\pi/\beta} \phi \ dx \ dy \left( \cdot \right)
\]

Boundary conditions:

$U = V = \phi = \frac{\partial \phi}{\partial z} = \psi = 0$ at $z = \pm 1$
Expansion series for wall direction

Boundary conditions:

\[ U = V = \phi = \frac{\partial \phi}{\partial z} = \psi = 0 \text{ at } z = \pm 1 \]

\[ F_\ell(\pm 1) = \frac{d}{dz} F_\ell(\pm 1) = G_\ell(\pm 1) = 0 \text{ for } \ell = 0, 1, 2, \cdots : \]

\[ F_\ell(z) = \frac{\ell + 1}{\ell + 2} T_{\ell+4}(z) - 2T_{\ell+2}(z) + \frac{\ell + 3}{\ell + 2} T_\ell(z), \]

\[ G_\ell(z) = \frac{T_\ell(z) - T_{\ell+2}(z)}{2} \]

\[ T_\ell(z) : \ell\text{-th Chebyshev polynomial for } \ell = 0, 1, 2, \cdots \]
Primary instability

Taylor number \( T \equiv \Omega (R - \Omega) \)

Critical state at \((T_c, \beta_c) = (106.74, 1.558)\)

Taylor vortex flow: \( T \geq T_c \)
Non-linear analysis: DNS

Expansions for $\phi$, $\psi$:

$$
\phi = \sum_{\ell=0}^{L} \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_{\ell mn}(t) F_\ell(z) \exp \left[ i (m\alpha x + n\beta y) \right], \\
\psi = \sum_{\ell=0}^{L} \sum_{m=-M}^{M} \sum_{n=-N}^{N} b_{\ell mn}(t) G_\ell(z) \exp \left[ i (m\alpha x + n\beta y) \right],
$$

where $a_{\ell00} = b_{\ell00} = 0$.

Expansions for mean flows $U$, $V$:

$$
U = \sum_{\ell=0}^{L} c_\ell(t) G_\ell(z), \quad V = \sum_{\ell=0}^{L} d_\ell(t) G_\ell(z) + \mu \left( z^2 - 1 \right).
$$

$\mu$ is determined such that there is no mean flow in span-wise ($y$) direction: $\int_{-1}^{1} V dz = 0$. 
Non-linear analysis: Equilibrium calculation

Expansions for $\phi$, $\psi$:

$$\phi = \sum_{\ell=0}^{L} \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_{\ell mn} F_{\ell}(z) \exp \left[ i \left( m\alpha(x - c t) + n\beta y \right) \right],$$

$$\psi = \sum_{\ell=0}^{L} \sum_{m=-M}^{M} \sum_{n=-N}^{N} b_{\ell mn} G_{\ell}(z) \exp \left[ i \left( m\alpha(x - c t) + n\beta y \right) \right],$$

where $a_{\ell00} = b_{\ell00} = 0$, and $c$ is a constant phase velocity.

Expansions for mean flows $U$, $V$:

$$U = \sum_{\ell=0}^{L} c_{\ell} G_{\ell}(z), \quad V = \sum_{\ell=0}^{L} d_{\ell} G_{\ell}(z) + \mu \left( z^2 - 1 \right).$$

$\mu$ is determined such that there is no mean flow in span-wise ($y$) direction: $\int_{-1}^{1} V \, dz = 0.$
Visualisation

Separation of flow field:

\[ u = (-R z + U(t, z)) i + V(t, z) j + \tilde{u}, \]
\[ \tilde{u} = \nabla \times (\nabla \times k \phi) + \nabla \times k \psi. \]

Moreover:

\[ \tilde{u} = \tilde{u} i + \tilde{v} j + \tilde{w} k = \nabla \times i \phi_1 + \nabla \times j \phi_2 + \nabla \times k \psi, \]

where

\[ \phi_1 = \frac{\partial \phi}{\partial y}, \quad \phi_2 = -\frac{\partial \phi}{\partial x}. \]

Each solenoidal potential can be a “stream function”:

\[ \phi_1 : \text{on } (y, z) \text{ plane for } \frac{\partial \tilde{u}}{\partial x} = 0, \]
\[ \phi_2 : \text{on } (z, x) \text{ plane for } \frac{\partial \tilde{v}}{\partial y} = 0, \]
\[ \psi : \text{on } (x, y) \text{ plane for } \frac{\partial \tilde{w}}{\partial z} = 0 \]
Linear stability analysis of non-linear states

Expansion for infinitesimal perturbation on non-linear state for $(R, \Omega; \alpha, \beta)$:

$$
\tilde{\phi}_\ell = e^{\sigma t} \sum_{\ell=0}^L \left[ \sum_{(m,n)\neq(0,0)} F_\ell(z) \tilde{a}_{\ell mn} \exp \left\{ i \left[ (m\alpha + d) (x - ct) + (n\beta + b) y \right] \right\} \right],
$$

$$
\tilde{\psi}_\ell = e^{\sigma t} \sum_{\ell=0}^L \left[ \sum_{(m,n)\neq(0,0)} G_\ell(z) \tilde{b}_{\ell mn} \exp \left\{ i \left[ (m\alpha + d) (x - ct) + (n\beta + b) y \right] \right\} \right],
$$

$\sigma$: Linear growth rate. Unstable mode: $(\alpha \pm d, \beta \pm b)$

$\tilde{\phi}$: Perturbation for non-linear state $\phi(R, \Omega; \alpha, \beta)$

$\tilde{\psi}$: Perturbation for non-linear state $\psi(R, \Omega; \alpha, \beta)$

$(d, b)$: Floquet parameters for a periodic state with $(\alpha, \beta)$

$\sigma = \sigma(d, b)$ for each nonlinear state $(\phi, \psi, U, V)$ for $(R, \Omega; \alpha, \beta)$

Stability boundary:

$$
\text{Re}[\sigma(d, b; R, \Omega, \alpha, \beta)] = 0 \quad \& \quad \partial \text{Re}(\sigma)/\partial d = \partial \text{Re}(\sigma)/\partial b = 0
$$
Stability boundary of Taylor vortex flow

\[ \beta = \beta_c (= 1.558) \]

\[ \text{M. Nagata: J Fluid Mech. 1988} \]
Stability boundaries

Figure 3 of Andereck et.al: Phys. Fluids, 26, 1395–1401 (1983)

(a) Taylor vortices
(b)
(c)
(d)
(e)
Stability boundary of TVF

$\beta = 1.25$

1. AZI
2. TWI, WVF
3. TWI+WIB
Instability of Taylor vortex flow

FIGURE 20. (a) Twisted Taylor vortices (TWI); $R_i = 1040$, $R_o = 720$. (b) Wavy inflow boundaries (WIB); $R_i = 1310$, $R_o = 700$. (c) Wavy outflow boundaries (WOB); $R_i = 1170$, $R_o = 700$. (d) Wavelets (WVL); $R_i = 1250$, $R_o = 730$. $f = 30$ for all four cases. The letters I and O indicate the inflow and outflow boundaries, respectively.

Fig. 20 of Andereck et.al: J. Fluid Mech. v. 164, pp. 155–183 (1986)
Instability boundary of Taylor vortex flow

- OTV
- WTV
- WVF
- O-WVF
- O-WVF (SUBH2)
Instability of Taylor vortex flow: $\beta = 1.25$

- Stability boundaries for Ordinary twist state

$$(R, \Omega, \alpha) = (150, 50, 1.7)$$
Instability of Taylor vortex flow: $\beta = 1.25$

- Stability boundaries for Wavy twist state

\[ (R, \Omega, \alpha) = (75, 65, 0.6) \]
Instability of Taylor vortex flow: $\beta = 1.25$

- Stability boundaries for Wavy vortex flow

$(R, \Omega, \alpha) = (110, 15, 0.38)$
Instability of Taylor vortex flow: $\beta = 1.25$

- Stability boundaries for Oscillatory Wavy vortex flow

$\text{(R, } \Omega, \alpha) = (180, 35, 0.27)$
Instability of Twist (Tertiary) state: $\beta = 1.25$

Fig. 9 ($\beta = 1.25$) of Hegseth's
Instability of Taylor vortex flow: $\beta = 1.25$

Fig. 20 of Andereck et.al: J. Fluid Mech. v. 164, pp. 155–183 (1986)
Instability of Twists (Tertiary) states: $\beta = 1.25$

- Stability boundaries (in blue) for quaternary states
- Most unstable $(d, b)$: subharmonics of twist states $(\alpha, \beta)$
**Wavy inflow/outflow boundary (WIB/WOB)**

- Subharmonic state bifurcated from Ordinary twist state (3x2)
- Phase unlocked drifting in $x$ independently

**WIB:** \( (R, \Omega, \alpha, \beta) = (130, 53, \beta_0/2, \beta_0/2) \) \( (\beta_0 = 1.25) \)

**WOB:** \( (\tilde{R}, \tilde{\Omega}, \alpha, \beta) = (130, 53, \beta_0/2, \beta_0/2) \) \( (\beta_0 = 1.25) \)
Characteristics of WIB/WOB

- Subharmonic state bifurcated from Ordinary twist state (3x2)
- Drifting in \( x \) (phase unlocked)
- Symmetry for Quaternary state WIB/WOB (\( Q^{WIB}/Q^{WOB} \)):

\[
Q^{WIB}(t, x - x_0, y - y_0, z) = Q^{WOB}(t, x_0 - x, y - y_0, -z)
\]

for coordinates \( x_0 \) and \( y_0 \).
- more symmetries for sub-solutions (like wave packets)

An example of the symmetry for coefficients. (\(*\): complex conjugate)

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>Poloidal</th>
<th>Toroidal</th>
<th>( U )</th>
<th>( V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>( a_{\ell m n}^{WOB} = (a_{\ell m n}^{WIB})^* )</td>
<td>( b_{\ell m n}^{WOB} = -(b_{\ell m n}^{WIB})^* )</td>
<td>( c_{\ell}^{WOB} = c_{\ell}^{WIB} )</td>
<td>0</td>
</tr>
<tr>
<td>even</td>
<td>( a_{\ell m n}^{WOB} = -(a_{\ell m n}^{WIB})^* )</td>
<td>( b_{\ell m n}^{WOB} = (b_{\ell m n}^{WIB})^* )</td>
<td>( c_{\ell}^{WOB} = -c_{\ell}^{WIB} )</td>
<td>0</td>
</tr>
</tbody>
</table>
Oscillatory wavy vortex flow (O-WVF)

- Subharmonic state bifurcated from Ordinary twist state (2x1)
- Oscillatory state with wavy twist symmetry
- \((R, \Omega) = (180, 52), (\alpha, \beta) = (0.9, 1.25)\)
Normalised (by $\tau_0 (= -R)$) momentum transport

TVF: $\beta = \beta_0$ ($\beta_0 = 1.25$)

OTV: $\alpha = \frac{3}{2} \beta_0$, $\beta = \beta_0$

quaternary WIB/WOB: $\alpha = \beta = \beta_0 / 2$

quaternary O-WVF: $\alpha = \frac{3}{4} \beta_0$, $\beta = \beta_0$

$\Omega = 54$
**Subharmonic instability of Taylor vortex flow: $\beta = 1.466$**

(Tertiary state: Subharmonic Oscillatory Wavy vortex flow)

- **WVL ($\beta = 1.466$)**
- $(R, \Omega) = (246, 66.2)$
  - Fig. 4c of Hegseth et.al: Phys. Rev. E, v. 53, pp. 507–521 (1996)

- $(R, \Omega) = (200, 60)$
- $(\alpha, \beta) = (0.612, 0.733)$
Appendix

Stability boundaries

(a) Taylor vortices

(b)

(c)

(d)

(e)

Figure 3 of Andereck et.al: Phys. Fluids, 26, 1395–1401 (1983)